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ON THE INTRODUCTION OF LORENTZ POLES INTO THE UNEQUAL-MASS SCATTERING AMPLITUDE

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Abstract

We suggest a new type of kinematical decomposition of the unequal-mass scattering amplitude. We introduce two non-commuting, non-disjunct Poincaré groups, P^+ and P^-, both of them are subgroups of the $P_1 \otimes P_2$ direct product group, where P_1 and P_2 are the Poincaré groups of the one-particle transformations for the first and the second particle of the two-particle states, respectively. The group P^+ is identical to the group of the two-particle Poincaré transformations. Our first decomposition for the scattering amplitude is a double expansion with respect to the representations of both the P^+ and P^- groups, simultaneously. The second proposal of ours is the partial wave analysis of not the center-of-mass states but of the "equal velocity" states, in which the individual particles move with the same velocity. Our expansions are valid for any s and t. In the equal-mass case they give the usual Lorentz pole decomposition at t=0. The formalism seems to be adequate for understanding the meaning of the "spectrum generating group" in the unequal-mass case. The variables of the expansion functions are unambiguously defined by the kinematical variables, and have branch-points only at the thresholds and pseudo-thresholds, in opposition to other approaches.

1. Introduction

During the last few years many attempts were made at eliminating the singularity developing at $t=0$ in the unequal-mass scattering amplitude when expanded in terms of Regge poles. Regarding the root of the problem two observations appeared to be important:

a/ The little-group of the two-particle total four-momentum contracts at zero energy;

b/ The center-of-mass system turns out to be meaningless at zero energy on mass-shell.

The first phenomenon expresses the fact that the little-group of lightlike four-vectors is essentially different from that of the timelike and spacelike ones. The second point means that the four-momentum of two particles having different masses can never be equal. Consequently, the partial wave expansion of the scattering amplitude in the center-of-mass frame cannot be used for analytic continuation to zero energy, and the singularity found there is not a singularity of the physical scattering amplitude, because we get out of the physical region of the four-momenta.

The solution of the problem is to suppose essentially the same situation as for equal-mass scattering. Namely, to suppose that there exist families of Regge poles gathered in irreducible representations of the Lorentz group. This was first noticed by Freedman and Wang [1] , a detailed analysis from group-theoretical point of view was given by Domokos and Tindle [2] , Domokos [3] , and by Toller and coworkers [4] . Another way to get rid of the singularity was found by di Vecchia et al. [5] and recently by Durand et al. [6] by using analytical methods.

Several attempts were made at using the notion of Lorentz poles at nonzero energy. Delbourgo, Salam and Strathdee published the first paper on it [7] , a different approach was elaborated by Domokos and Surányi [8] , and later by Toller [9] . The analytical methods are powerful enough for this purpose as well.

Let alone the analytical approach, in the others either off-mass shell amplitudes were necessary or there were problems with momentum conservation. In our opinion the origin of these problems is that none of these approaches exhausts maximally the informations which the two-particle states involve from the point of view of Lorentz group. In the present paper we give a detailed account of two-particle kinematics and give the possible group-theoretical descriptions of two-particle states. Based on this review we give expansions for the two-particle scattering amplitude which explicitly make use of Lorentz invariance. Our reason for giving these very complicated expansions is that, in our opinion, these expansions lead in the most natural manner to the group-theoretical introduction of the Lorentz poles into the scattering amplitude in the unequal-mass case. From this point of view this paper can be considered a general frame to approach the Lorentz pole problem, and further investigations are necessary to make explicit the implications of the programme.

2.1. The two-particle states

First we wish to give a detailed description of two-particle states from the point of view of Poincaré representations.

The two-particle states are the elements of a linear space defined as the direct product space of one-particle states. The most usual and simplest way to enumerate the vectors of the direct product space is to enumerate them for both particles separately. A two-particle state, denoted as $|p_1, s_1, \lambda_1 ; p_2, s_2, \lambda_2 >$, has twelve indices, namely, the four-momenta p_1 and p_2 , the spins s_1 and s_2 , and the helicities λ_1 and λ_2 . Over this space the Poincaré transformations are generated by twice ten operators P'_μ , $M'_{\mu\nu}$, P''_μ , $M''_{\mu\nu}$. P'_μ and $M'_{\mu\nu}$ are the four-momentum and angular momentum operators, respectively, for the particle 1, the double primed operators are the same for the particle 2. The former representation diagonalizes the P'_μ and P''_μ operators. The indices s_1 and s_2 are the eigenvalues of the Casimir operators $W'_\mu W'_\mu$ and $W''_\mu W''_\mu$ where

$$W_\mu = \frac{1}{2}\, \varepsilon_{\mu\nu\rho\kappa}\, M_{\nu\rho}\, P_\kappa \,. \tag{2.1}$$

The helicities λ_1 and λ_2 are the eigenvalues of W'_o and W''_o . The irreducibility of the representation space manifests itself in the fact that beside s_1 and s_2 the eigenvalues of the other two Casimir operators, $P'_\mu\ P'_\mu$ and $P''_\mu\ P''_\mu$, are also fixed: $p_1^2 = m_1^2$ and $p_2^2 = m_2^2$. There is one more invariant quantity for the Poincaré group: the sign of the eigenvalues of P_o . We choose, by convention, this sign to be positive for both particles.

This set of quantum numbers is not a practical one when we want to exploit Poincaré invariance. The inhomogeneous Lorentz transformations are primarily the transformations of space-time, consequently, of the two-particle system as a whole. These transformations are generated by the operators

$$P_\mu^{(+)} = P'_\mu + P''_\mu \quad , \quad M_{\mu\nu}^{(+)} = M'_{\mu\nu} + M''_{\mu\nu} \quad . \tag{2.2}$$

These operators form a Poincaré subalgebra of the direct sum algebra of P'_μ , P''_μ , $M'_{\mu\nu}$, $M''_{\mu\nu}$. Since p_1 and p_2 have not simple transformation properties under the transformations of the Poincaré subgroup generated by the 2.2 operators, they are usually changed to s , $\underline{p}^{(+)}$, $W^{(+)}$, $m^{(+)}$, to the eigenvalues of the operators $P_\mu^{(+)} P_\mu^{(+)}$, $P_i^{(+)}$, $W_\mu^{(+)} W_\mu^{(+)}$, $W_o^{(+)}$ or $W_3^{(+)}$, respectively. The usefulness of s and $W^{(+)}$ is obvious. The aim of diagonalizing the operators $P_i^{(+)}$ and $W_o^{(+)}$ is to have simple representation for the translation operators. In some cases, however, this choice is not the most advantageous one, it is better to diagonalize the operators

$$\frac{1}{4}\, \varepsilon_{\mu\nu\rho\kappa}\, M_{\mu\nu}^{(+)} M_{\rho\kappa}^{(+)} \;,\quad \frac{1}{2} M_{\mu\nu}^{(+)} M_{\mu\nu}^{(+)} \;,\quad M_i^{(+)} M_i^{(+)} \;,\quad M_3^{(+)} \;, \tag{2.3}$$

where $M_i^{(+)} = \frac{1}{2}\, \varepsilon_{ijk}\, M_{kj}^{(+)}$.

This choice is built upon the $SL(2,C)$ part of the Poincaré subalgebra of the generators (2.2) . In this case we use instead of $\underline{p}^{(+)}$ and $m^{(+)}$ the quantum numbers j_o , σ , j , m defined by the eigenvalues of the operators (2.3) : $ij_o\sigma$, $j_o^2 - \sigma^2 + 1$, $j(j+1)$, m , respectively. One more generalization can be done in choosing the set (2.3) , namely, one can use the Casimir operator of other subgroups of $SL(2,C)$ instead of $M_i^{(+)} M_i^{(+)}$ (c.f. Appendix A). One problem arises from the change of $\underline{P}^{(+)}$, $W_o^{(+)}$ to the set (2.3) : W'_o and W''_o do not commute with all the operators of this set. However, the operators $(W'_\mu + W''_\mu)^2$, $P_\mu^{(+)}(W'_\mu - W''_\mu)$ make again complete the set of the 12 commuting operators. We note here only that they are "Lorentz invariant" operators, that is, they commute with the generators $M_{\mu\nu}^{(+)}$, further details will be given later. We shall denote the eigenvalues of $P_\mu^{(+)}(W'_\mu - W''_\mu)$ and $(W'_\mu + W''_\mu)^2$ with $\lambda^{(+)}$ and $\Sigma^{(+)}$, respectively.

Now let $\Lambda^{(+)}$ be an element of the homogeneous Lorentz group generated by the $M_{\mu\nu}^{(+)}$ operators, and let $U(\Lambda^{(+)})$ be the operator which represents it on the space spanned by the basis vectors $| m_1 , m_2 , s_1 , s_2 ; s , W^{(+)}, j_o , \sigma , j , m ; \Sigma^{(+)}, \lambda^{(+)} >$. (In the following the indices m_1 , m_2 , s_1 , s_2 will be suppressed.) Obviously,

$$U(\Lambda^{(+)})|\ldots; j_o, \sigma, j, m; \ldots\rangle = \sum_{j', m'} D^{j_o \sigma}_{j'm'jm}(\Lambda^{(+)})|\ldots; j_o, \sigma, j', m'; \ldots\rangle , \qquad (2.4)$$

where $D^{j_o\sigma}_{j'm'jm}$ is an $SL(2,C)$ representation matrix element, well-known from the literature [14, 15, 16, 17].

There exists another possible choice of quantum numbers and it is based on the fact that one can find a Poincaré subgroup different from the previous one. It is generated by the operators:

$$P^{(-)}_o = P'_o - P''_o , \quad P^{(-)}_i = P'_i + P''_i , \quad N^{(-)}_i = N'_i - N''_i , \quad M^{(-)}_i = M'_i + M''_i . \qquad (2.5)$$

Here $N_i = M_{oi}$. We shall call this subgroup the $P^{(-)}$ group, contrary to the previous $P^{(+)}$ group. An element of its homogeneous part, generated by the $M^{(-)}_i$ and $N^{(-)}_i$ operators, will be denoted by $\Lambda^{(-)}$. It is evident that we can define the analogue of the, former $(+)$ -type commuting operator systems changing the $(+)$ -type generators to $(-)$ -type ones. Denoting the new quantum numbers with τ, $W^{(-)}$, ℓ_o, ρ, ℓ, μ, $\Sigma^{(-)}$, $\lambda^{(-)}$, we can write the analogue of (2.4) as follows:

$$U(\Lambda^{(-)})|\ldots; \ell_o, \rho, \ell, \mu; \ldots\rangle = \sum_{\ell', \mu'} D^{\ell_o \rho}_{\ell'\mu', \ell\mu}(\Lambda^{(-)})|\ldots; \ell_o, \rho, \ell', \mu'; \ldots\rangle . \qquad (2.6)$$

Obviously, the explicit form of the functions $D^{\ell_o\rho}_{\ell'\mu', \ell\mu}$ is the same as that of the functions $D^{j_o}_{j'm', jm}$.

Now we summarize the commuting operator sets and quantum numbers we have spoken about:

a) $\begin{array}{llllll} P'_\mu P'_\mu , & W'_\mu W'_\mu , & P''_\mu P''_\mu , & W''_\mu W''_\mu ; & \underline{P}', W'_o , & \underline{P}'', W''_o \\ m_1 & s_1 & m_2 & s_2 & \underline{p}_1, \lambda_1 , & \underline{p}_2, \lambda_2 \end{array}$

b) $\begin{array}{llll} P^{(+)}_\mu P^{(+)}_\mu , & W^{(+)}_\mu W^{(+)}_\mu , & \underline{P}^{(+)}, W^{(+)}_o ; & W'_o , W''_o \\ s & W^{(+)} & \underline{p}^{(+)}, m^{(+)} ; & \lambda_1 , \lambda_2 \end{array}$

c) $\begin{array}{llll} P^{(-)}_\mu P^{(-)}_\mu , & W^{(-)}_\mu W^{(-)}_\mu , & \underline{P}^{(-)}, W^{(-)}_o ; & W'_o , W''_o \\ \tau & W^{(-)} & \underline{p}^{(-)}, m^{(-)} ; & \lambda_1 , \lambda_2 \end{array}$

d) $\begin{array}{llllllll} P^{(+)}_\mu P^{(+)}_\mu , & W^{(+)}_\mu W^{(+)}_\mu , & \underline{M}^{(+)2} - \underline{N}^{(+)2} , & \underline{M}^{(+)}\underline{N}^{(+)} , & \underline{M}^{(+)2} , & M^{(+)}_3 , & (W'_\mu + W''_\mu)^2 , & P^{(+)}_\mu (W'_\mu - W''_\mu) \\ & & j_o , & \sigma , & j , & m , & \Sigma^{(+)} & \lambda^{(+)} \end{array}$

e) $\begin{array}{llllllll} P^{(-)}_\mu P^{(-)}_\mu , & W^{(-)}_\mu W^{(-)}_\mu , & \underline{M}^{(-)2} - \underline{N}^{(-)2} , & \underline{M}^{(-)}\underline{N}^{(-)} , & \underline{M}^{(-)2} , & M^{(-)}_3 , & (W'_\mu + g_{\mu\nu} W''_\nu)^2 , & P^{(-)}_\mu (W'_\mu - g_{\mu\nu} W''_\nu) . \\ \tau & & \ell_o , & \rho , & \ell , & \mu , & \Sigma^{(-)} , & \lambda^{(-)} \end{array}$

Concerning the quantum numbers we make the following remarks:

1. As a consequence of the fact that the $P^{(+)}$ and $P^{(-)}$ algebras are not disjunct (because $\underline{P}^{(+)} = \underline{P}^{(-)}$, $\underline{M}^{(+)} = \underline{M}^{(-)}$), the quantum numbers $\underline{p}^{(+)}$, $\underline{p}^{(-)}$ and j , m , ℓ , μ are the same in the sets b), c) and d), e), respectively. Working with subgroups of $SL(2,C)$ different from that of the M_i -s, the appropriate indices obviously lose this property.

2. A $\Lambda^{(+)}$ transformation leaves invariant the form $(p_o' + p_o'')^2 - (\underline{p}' + \underline{p}'')^2$, whereas $\Lambda^{(-)}$ does the same with $(p_o' - p_o'')^2 - (\underline{p}' + \underline{p}'')^2$. The notions "scalar", "vector" etc. are different for the $P^{(+)}$ and $P^{(-)}$ groups.

3. In the sets a), b), c) instead of λ_1 and λ_2 one may use the quantum numbers $\Sigma^{(+)}$, $\lambda^{(+)}$ or $\Sigma^{(-)}$, $\lambda^{(-)}$.

4. It is a highly delicate question how to transform from one type of basis system to another one. This problem will be treated in the Appendices.

Finishing the discussion of the quantum numbers we deal with the eigenvalues of the operators $(W_\mu' + W_\mu'')^2$ and $P_\mu^{(+)}(W_\mu' - W_\mu'')$. We can write on momentum states:

$$W_\mu |p,s,\lambda > = \frac{1}{2}\, \varepsilon_{\mu\nu\rho\kappa}\, M_{\nu\rho}\, p_\kappa\, |p,s,\lambda > \equiv S_\mu(p)\, |p,s,\lambda > . \tag{2.7}$$

The transformation property of the operator $S_\mu(p)$ under Lorentz transformations is:

$$U(\Lambda)\; S_\mu(p)\; U^{-1}(\Lambda) = L_{\mu\nu}(\Lambda)\; S_\nu(\Lambda p) \quad . \tag{2.8}$$

The notation is obvious. Since $(W_\mu' + W_\mu'')^2$ and $P^{(+)}(W_\mu' - W_\mu'')$ are covariant operators we may confine ourselves to their eigenvalues on "equal velocity states" (the particles have the same velocity of opposite direction; see Section 3.) . We use the phase convention of Jacob and Wick for two-particle states [11] and write :

$$|p_1,s_1,\lambda_1;p_2,s_2,\lambda_2 > = B_1(\alpha)\; R_2(\pi,\pi,0)\; B_2(\alpha)\, |m_1,s_1,\lambda_1 > \otimes\; (-1)^{s_2-\lambda_2}|m_2,s_2,\lambda_2>$$

Here B and R are boost and rotation operators acting on one-particle states. Let us introduce the following linear combination:

$$|p_1,s_1,p_2,s_2;\sigma,\lambda > = \sum_{\lambda_1,\lambda_2} C^{\sigma\lambda}_{s_1\lambda_1,s_2-\lambda_2}\, |p_1,s_1,\lambda_1;p_2,s_2,\lambda_2 > . \tag{2.9}$$

The linear combination is made by the Clebsch-Gordan coefficients of the rotation group. These states are eigenstates of the operator $P_\mu^{(+)}(W_\mu' - W_\mu'')$.

Using eqs. (2.7) and (2.8) we find:

$$P_\mu^{(+)}(W'_\mu - W''_\mu)|p_1,s_1,p_2,s_2;\sigma,\lambda> =$$

$$= \frac{\lambda}{2}\left[s^2 - 2s(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2\right]^{1/2} |p_1,s_1,p_2,s_2;\sigma,\lambda> \quad .$$

The situation is a bit more complicated in the case of $(W'_\mu + W''_\mu)^2$. After some straightforward calculation we get the following result:

$$(W'_\mu + W''_\mu)^2 \;|p_1,s_1,p_2,s_2;\sigma,\lambda> = \Big\{\big[m_1(m_1 - m_2)\; s_1(s_1+1) -$$

$$-m_2(m_1 - m_2)\; s_2(s_2+1) + m_1 m_2\; \sigma(\sigma+1)\big]\delta_{\sigma\sigma'} + \qquad (2.10)$$

$$+ \left[2m_1 m_2 - \frac{\Delta(s,m_1^2,m_2^2)}{2s}\right] \sum_{\lambda_1\lambda_2} C^{\sigma\lambda}_{s_1\lambda_1,s_2-\lambda_2}\; C^{\sigma'\lambda}_{s_1\lambda_1,s_2-\lambda_2}\; \lambda_1\lambda_2\Big\}|p_1,s_1,p_2,s_2;\sigma',\lambda>.$$

This result says that there is a one-to-one correspondence between the "total spin" values σ and the eigenvalues of the operator $(W'_\mu + W''_\mu)^2$, and its eigenstates are linear combinations of the states defined by eq. (2.9) :

$$|p_1,s_1,p_2,s_2;\Sigma,\lambda> = \sum_\sigma A_{\Sigma\sigma}\; |p_1,s_1,p_2,s_2;\sigma,\lambda> \;,$$

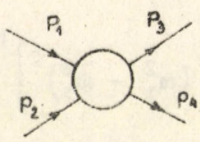
$$(W'_\mu + W''_\mu)^2|p_1,s_1,p_2,s_2;\Sigma,\lambda> \sim |p_1,s_1,p_2,s_2;\Sigma,\lambda> \;. \qquad (2.11)$$

We mention that in practical cases (e.g. $s_1 = 0$, s_2 arbitrary; $s_1=s_2=\frac{1}{2}$; $p_1=p_2=0$) no diagonalization is necessary.

Similar results can be found by repeating the calculations for the operators $(W'_\mu + g_{\mu\nu} W''_\nu)^2$, $P_\mu^{(-)}(W'_\mu - g_{\mu\nu}W''_\nu)$. In the following, either we are working with a (+) -type set of quantum numbers or with a (-) -type one, we shall use the symbols Σ and λ , omitting the $(\pm)$ indices. However, we must not forget that there is no diagonality between the $\Sigma^{(+)}$ and $\Sigma^{(-)}$ values, while $\lambda^{(+)}$ and $\lambda^{(-)}$ are essentially the same (c.f. Appendix C.) .

3.1. The expansion of the scattering amplitude

After these preliminary steps now we concentrate to our very problem, to the expansion of the scattering amplitude. Let us consider the scattering process drawn in Fig. 1. We should like to treat it at high values of s , assuming exchanged poles in the t -channel. Owing to crossing symmetry we have the following connection between the s and t -channel scattering amplitude in center-of-mass system [11] :



$$s = (p_1 + p_2)^2$$

$$t = (p_1 - p_3)^2$$

Fig. 1.

$$\langle p_1, s_1, \lambda_1; p_2, s_2, \lambda_2 | T | p_3, s_3, \lambda_3; p_4, s_4, \lambda_4 \rangle =$$

$$= \sum_{\lambda_i'} d^{s_1 *}_{\lambda_1' \lambda_1} d^{s_2 *}_{\lambda_2' \lambda_2} d^{s_3}_{\lambda_3' \lambda_3} d^{s_4}_{\lambda_4' \lambda_4} < p_1^c, s_1, \lambda_1'; p_3^c, s_3, \lambda_3' | T | p_4^c, s_4, \lambda_4'; p_2^c, s_2, \lambda_2' > ,$$

where $(p_1 + p_2)^2 = s$, $(p_1 - p_3)^2 = t$, $\underline{p}_1 + \underline{p}_2 = 0$,

$(p_1^c + p_3^c)^2 = t$, $(p_1^c - p_2^c)^2 = s$, $\underline{p}_1^c + \underline{p}_3^c = 0$.

Here and in the following when writing matrix elements like $< p_1 , s_1 , \lambda_1 , p_2 , s_2 , \lambda_2 | T | p_3 , s_3 , \lambda_3 ; p_4 , s_4 , \lambda_4 >$ we always think of four-momenta which satisfy momentum conservation, but we shall never write explicitly the δ -function $\delta^4(p_1 + p_2 - p_3 - p_4)$.

Instead of expanding $f^s = <p_1 , s_1 , \lambda_1 ; p_2 , s_2 , \lambda_2 |T| p_3 , s_3 , \lambda_3 ; p_4 , s_4 , \lambda_4>$ in the crossed channel we shall do the same with $f^t = <p_1^c, s_1, \lambda_1 ; p_3^c, s_3, \lambda_3 |T| p_4^c, s_4 , \lambda_4 ; p_2^c , s_2 , \lambda_2 >$ in the direct one. First we shall consider f^t in its physical domain, then we continue it analytically in s and t to the physical region of f^s. At the end of this procedure one particle is negative timelike on both sides of T in the two-particle states. Toller showed [4] that such a functional does not exist everywhere in the crossed domain though it is dense in it. This means that in a more rigorous treatment our formulae ought to be considered as tools to define a functional, the domain of definition of which can be extended to the whole region in question. However, our way of speaking is generally accepted in the physical literature, see e.g. ref. [12] and many others.

Throughout the analytic continuation the kinematical singularities could cause trouble, so we get rid of them multiplying f^t with an appropriate $K(s,t)$ function. Its form is well-known for any definite process [13].

$$\bar{f}^t = K(s,t)\ f^t \quad . \tag{3.1.2}$$

In what follows we shall speak only about the right-hand-side ket of f^t to save place, but we always mean the whole amplitude. The right-hand-side ket can be written as follows:

$$|p_4,s_4,\lambda_4;p_2,s_2,\lambda_2>_{C.M.} = L'(p_4)\,|m_4,s_4,\lambda_4\rangle \otimes L''(p_2)\,(-1)^{s_2-\lambda_2}|m_2,s_2,\lambda_2> =$$

$$= e^{-i\phi M_3^{(+)}}\, e^{-i\Theta M_2^{(+)}}\, e^{-i\alpha_4 N_3'+i\alpha_2 N_3''}$$

$$\times\, e^{-i\pi M_2''}\, e^{i\phi M_3^{(+)}}\, |m_4,s_4,\lambda_4\ ;\ m_2,s_2,\lambda_2> \tag{3.1.3}$$

We introduce the notation

$$e^{-i\pi M_2''}\, e^{i\phi M_3^{+}}\, |m_4,s_4,\lambda_4> \otimes (-1)^{s_2-\lambda_2}\, |m_2,s_2,\lambda_2> = \tag{3.1.4}$$

$$= e^{i\phi(\lambda_1+\lambda_2)}\, |m_4,s_4,\lambda_4> \otimes |m_2,s_2,-\lambda_2> \equiv |R> \quad , \tag{3.1.5}$$

moreover, we may write

$$e^{-i\alpha_4 N_3'+i\alpha_2 N_3''} = e^{-i\alpha N_3^{(+)}}\, e^{-i\beta N_3^{(-)}}$$

where $N_3^{(+)} = N_3' + N_3''$, $N_3^{(-)} = N_3' - N_3''$.
Now we can write $(3.1.3)$ in the following form:

$$|p_4,s_4,\lambda_4;\ p_2,s_2,\lambda_2>_{C.M.} = e^{-i\phi M_3^{(+)}}\, e^{-i\Theta M_2^{(+)}}\, e^{-i\alpha N_3^{(+)}}\, e^{-i\beta N_3^{(-)}}\, |R> . \tag{3.1.6}$$

Since any two-particle state appears like $\Lambda^{(+)}|p_1,\ s_1,\ \lambda_1\ ;\ p_2,s_2,\lambda_2>_{C.M.}$, it is also true due to $(3.1.6)$ that any two-particle state can be written as $\Lambda^{(+)} B^{(-)}|R>$, where $\Lambda^{(+)}$ is a general Lorentz transformation of $(+)$-type with six parameters (the last rotation around the z-axis gives only a phase, five parameters are essential), $B^{(-)}$ is a boost of $(-)$-type with one parameter, wich gives the particles velocity in the z-direction. Consequently, the state $\exp(-i\beta N_3^{(-)})|R>$ is a good standard state for two-particle states in the sense that any other state can be obtained from it by applying Lorentz transformation. Nay, it is a better standard state than the center-of-mass state, because we have trouble with the latter at zero energy. Another aspect of eq. $(3.1.6)$ is that the two-particle states can be considered as a function over both the $P^{(+)}$ and $P^{(-)}$ groups. Our expansion for the scattering amplitude will be a simultaneous expansion with respect to the representation functions of these groups.

One can easily check from (3.1.5) that

$$\cosh\beta = \frac{1}{2\sqrt{m_2 m_4}}\sqrt{t-(m_2-m_4)^2} \qquad \cosh\alpha = \frac{m_2+m_4}{2\sqrt{t m_2 m_4}}\sqrt{t-(m_2-m_4)^2}\ . \tag{3.1.7}$$

Hence

$$\beta = \ln\frac{1}{2\sqrt{m_2 m_4}}\left[\sqrt{t-(m_2-m_4)^2}+\sqrt{t-(m_2+m_4)^2}\right] \tag{3.1.8}$$

and there is a similar expression for α . (For definitness we suppose $m_4 \geq m_2$) . Now we continue analytically in the variable t. The path of continuation is drawn on Fig. 2. We start from $t' + i\varepsilon$, $t' > (m_2+m_4)^2$ and go to $t'' - i\varepsilon$, $0 < t'' < (m_2-m_4)^2$, $\varepsilon > 0$. For the time being we do not go to smaller than zero values of t , because singularities appear in α at $t = 0$. During the continuation t is real somewhere, and we choose this point to be between the threshold and pseudo-threshold [11] . A glance at eq. (3.1.8) shows that during the continuation we cross the cut of the ln -function, and we cross the cut of one of the square-roots, hence their relative sign alters. Consequently, at the end of the continuation path β becomes $\beta' + i\frac{\pi}{2}$ where

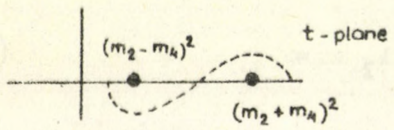


Fig. 2.

$$\cosh\beta' = \frac{1}{2\sqrt{m_2 m_4}}\sqrt{(m_2+m_4)^2 - t}\ . \tag{3.1.9}$$

Similarly, in the case of α , we get a value $\alpha' + i\frac{\pi}{2}$ at the end of the continuation path with

$$\cosh\alpha' = \frac{m_4-m_2}{2\sqrt{t m_2 m_4}}\sqrt{(m_2+m_4)^2 - t}\ . \tag{3.1.10}$$

We shift the effect of the $i\frac{\pi}{2}$ angles to the state $|R\rangle$. Let alone a phase factor it will alter the sign of the crossed particle and does nothing else. Denoting $\exp(-i\pi N_3'')|R\rangle$ with $|\bar{R}\rangle$ we obtain the following expression for the scattering amplitude:

$$f^t_{\lambda_1\lambda_2\lambda_3\lambda_4} = \langle \bar{R}'| e^{i\beta'' N_3^{(-)}} e^{+i\alpha'' N_3^{(+)}} T e^{-i\Theta M_2^{(+)}} e^{-i\alpha' N_3^{(+)}} e^{-i\beta' N_3^{(-)}} |\bar{R}\rangle , \qquad (3.1.11)$$

where β'' and α'' can be obtained from β' and α' by changing m_1 to m_4 and m_2 to m_3 , Θ is the t -channel scattering angle:

$$\cos\Theta = \frac{t(s-u) + (m_1^2-m_3^2)(m_2^2-m_4^2)}{\left\{\left[t-(m_1-m_3)^2\right]\left[t-(m_1+m_3)^2\right]\left[t-(m_2-m_4)^2\right]\left[t-(m_2+m_4)^2\right]\right\}^{1/2}} .$$

Assuming that $[T,M_{\mu\nu}] = 0$, the (+) -type transformations can be added [15] :

$$e^{i\alpha'' N_3^{(+)}} e^{-i\Theta M_2^{(+)}} e^{-i\alpha' N_3^{(+)}} = e^{-i\psi M_2^{(+)}} e^{-i\xi N_3^{(+)}} e^{-i\chi M_2^{(+)}},$$

where

$$\cosh\xi = \{s(m_1m_2+m_3m_4) - u(m_1m_3+m_2m_4) + (m_1-m_3)(m_2-m_4)(m_1m_3+m_2m_4)\} \times$$

$$\times \{4m_1m_2m_3m_4 \left[(m_1+m_3)^2 - t\right]\left[(m_2+m_4)^2 - t\right]\}^{-1/2} ,$$

$$\frac{1}{\cos\psi} = -\sinh\xi\,(\cosh\alpha' \sinh\alpha'' - \cos\Theta \sinh\alpha' \cosh\alpha'')^{-1} \qquad (3.1.12)$$

$$\frac{1}{\cos\chi} = \sinh\xi\,(\cosh\alpha'' \sinh\alpha' - \cos\Theta \sinh\alpha'' \cos\alpha')^{-1} .$$

These formulae are analytic expressions of s and t with singularities only at the thresholds and pseudo-thresholds, even the pathological singularity at $t = 0$ disappeared.

Now we are in the position to give an expansion of the scattering amplitude in the following manner: we insert complete systems of states labelled by sets of quantum numbers of types d) and e) of Sect. 2. into the expression (3.1.11) for the scattering amplitude:

$$f^t_{\Sigma\lambda,\Sigma^*\lambda^*}(s,t) = \sum \langle m_1,s_1,-m_3,s_3;\Sigma,\lambda|(-)\rangle\langle(-)| e^{i\beta'' N_3^{(-)}} |(-)\rangle\langle(-)|(+)\rangle \times$$

$$\times \langle(+)| e^{-i\psi M_2^{(+)}} e^{-i\xi N_3^{(+)}} e^{-i\gamma M_2^{(+)}} |(+)\rangle\langle(+)|T|(+)\rangle\langle(+)|(-)\rangle \times$$

$$\times \langle(-)| e^{-i\beta' N_3^{(-)}} |(-)\rangle\langle(-)|m_4,s_4,-m_2;\Sigma^*,\lambda^*\rangle .$$

Here we changed the labels λ_1 , λ_2 to Σ and λ in the $|R\rangle$ states. We get the final form of this expansion making use of Poincaré invariance and the Wigner-Eckart theorem, moreover of (2.4) and (2.6):

$$f^{t}_{\Sigma\lambda,\Sigma^*\lambda^*}(s,t) = \sum \langle m_1,s_1,-m_3,s_3;\Sigma,\lambda|(m_1+m_3)^2,\ W^{(-)};\ell_o,\rho,\ell,\mu;\Sigma,\lambda\rangle \times$$

$$\times\ d^{\ell_o\rho}_{\ell\mu\ell'}(-\beta'')\langle(m_1+m_3)^2,\ W^{(-)};\ell_o,\rho,\ell',\mu;\Sigma,\lambda|t,W^{(+)};j_o,\sigma,j,m;\Sigma',\lambda'\rangle \times$$

$$\times\, T_{\Sigma',\lambda';\Sigma'',\lambda''}(t,W^{(+)})\ D^{j_o\sigma}_{jm,j'm'}(\psi,\xi,\chi) \times \tag{3.1.13}$$

$$\times\ \langle t,W^{(+)};j_o,\sigma,j',m';\Sigma'',\lambda''|(m_2+m_4)^2,\ W^{(-)\prime};\ell_o',\rho',\ell'',\mu'';\Sigma^*,\lambda^*\rangle \times$$

$$\times\ d^{\ell_o'\rho'}_{\ell''\mu''\ell''}(\beta')\ \langle(m_2+m_4)^2,\ W^{(-)\prime};\ \ell_o',\rho',\ell'',\mu'';\Sigma^*,\lambda^*|m_4,s_4,-m_2,s_2;\Sigma^*,\lambda^*\rangle .$$

Another expansion can also be given, which does not involve basis states of (-) type at all. Instead we expand directly the states $\exp\left(-i\,\beta\, N_3^{(-)}\right)|R\rangle$ with respect to states of (+) type. The states $\exp(-i\beta N_3^{(-)})|m_1;s_1;\lambda_1;m_2,s_2,\lambda_2\rangle \equiv$ $\equiv |m_1,s_1,\lambda_1;\ m_2,s_2,\lambda_2;\ v\rangle$, which we call "equal velocity states", can easily be expanded with respect to a basis of type d) of Sect.2., except that one must apply for labeling the states the Casimir operator eigenvalue of the appropriate "interpolating group", that is, of the little-group corresponding to the total four-momentum of the state $\exp(-i\ \beta\ N_3^-)|m_1,s_1,\lambda_1;\ m_2,s_2,\lambda_2\rangle$, instead of the eigenvalue of $\underline{M}^2$, the Casimir operator of the SU(2) subgroup of SL(2,C). Further details of this point are given in the Appendices A, B. The final result is the following expansion of the "equal velocity states":

$$|m_1,s_1,m_2,s_2;\Sigma,\lambda;v\rangle = \sum N(t,W;j_o,\sigma)\ \delta_{W^{(+)}k}\ \delta_{\lambda m}|t,W^{(+)};j_o,\sigma,k,m;v;\Sigma,\lambda\rangle . \tag{3.1.14}$$

The notation $\delta_{kW^{(+)}}$ expresses the fact that for the "equal velocity" states" the connection

$$W^{(+)2} = p_o^2\left[k^2 - \frac{1}{4}(1 - v^2)\right]$$

exists between the indices $W^{(+)}$ and k of the states which appear in the sum on the right-hand-side of (3.1.13) . v is defined as $v = \tanh\beta$, and the total four-momentum of the "equal velocity states" is $p_o(1,0,0,v)$, $t = p_o^2(1 - v^2)$. $N(t,\ W^{(+)},\ j_o,\ \sigma)$ is a normalization factor. The expansion of the scattering amplitude is as follows:

$$f^{t}_{\Sigma,\lambda;\Sigma',\lambda'}(s,t) = \sum N(t,W^{(+)};j_o,\sigma)\ N^*(t,W^{(+)};j_o,\sigma)\ \delta_{kW^{(+)}}\delta_{\lambda\mu'}\delta_{\lambda'\mu''} \times$$

$$\times\ \langle t,W^{(+)};j_o,\sigma,k',\mu';0;\Sigma,\lambda|t,W^{(+)};j_o,\sigma,k,\mu';v;\Sigma,\lambda\rangle \times \tag{3.1.15}$$

$$T_{\Sigma,\lambda;\Sigma',\lambda'}(t,W^{(+)})\ D^{j_o\sigma}_{k'\mu',k''\mu''}(\Lambda^{(+)})\ \langle t,W^{(+)},j_o,\sigma,k,\mu'';v;\Sigma',\lambda'|t,W^{(+)};j_o,\sigma,k'',\mu'';0;\Sigma',\lambda'\rangle.$$

To evade the problem of evaluating matrix elements like

$$<j_o,\sigma,k,m;v|U(\Lambda^{(+)})|j_o,\sigma,k',m';v>$$

we have inserted complete systems of states $|j_o, \sigma, k, m; 0>$, which are the old-fashioned SU(2) basis states for the SL(2,C) representations. Then we are faced the problem of the evaluation of the transformation coefficients $<j_o, \sigma, k', m; v|j_o, \sigma, k, m; 0>$. There are standard methods for solving this problem, one of them is outlined in Appendix A.

3.2. Discussion

In the following we wish to discuss some crucial points of the previous paragraph.

First we examine the problem of Poincaré invariance in the crossed channel. We have seen that two Poincaré subgroups are involved in the Poincaré $\otimes$ Poincaré direct product group, which is represented on the space of the two-particle states, and one must answer the question which of them is the invariance group. The difficulty with answering this question arises, because we do not know the "translation" of the physical space-time transformations to the "language" of the transformations of the non-physical crossed states. Consequently, we must look for indirect solution of the problem.

It is obvious from analyticity requirements that we must maintain the validity of $[T, M^{(+)}_{\mu\nu}]=0$ even in the crossed channel. Let us assume that in the crossed channel we have from Lorentz invariance the relation $[T,M^{(-)}_{\mu\nu}]=0$ and $[T,M^{(+)}_{\mu\nu}]=0$ is an additional symmetry of the T scattering operator. Then it follows that the $P^{(-)}$ group is the fundamental invariance group. However, it is easy to see that these assumptions make T a zero operator. The first difficulty is that now there is not diagonality in the quantum numbers t and $W^{(+)}$. The problem cannot be solved by enlarging the "additional symmetry group" to the whole $P^{(+)}$ group, because now t is not a "Lorentz invariant" quantum number, neither $W^{(+)}$. We might say that there is not diagonality in the variables t and $W^{(+)}$, but only the diagonal matrix elements have physical meaning. Even this very strange assumption does not solve anything, for we are faced still now the trouble that the Casimir operators of the $P^{(-)}$ group are not invariant with respect to the $\Lambda^{(+)}$ transformations, either. The problem is unchanged if we weaken the assumption $[T,M^{(+)}_{\mu\nu}]=0$ and say that it is not true like an operator relation but only between lightlike states. (This would correspond to the "additional symmetry" of T only at $t=0$) . Finally, we conclude that the relation $[T,M^{(+)}_{\mu\nu}] = 0$ is valid and expresses the Lorentz invariance in both the direct and crossed channels. (At an earlier stage of this work we tried to answer this question studying the behaviour of α_2 and α_4 during the analytic continuation from the t -channel to the s -channel, and concluded that $[T,M^{(+)}_{\mu\nu}]=0$ was a new condition for T to maintain analyticity at $t = 0$. It is clear from the previous discussion that this conclusion was wrong.)

The second question we must answer: What kind of Poincaré and Lorentz group representations appear in the expansions (3.1.13) and (3.1.15) ? In other words, we must specify what states are contained in our "complete systems of states". To give a precise answer first of all we ought to know the function space which the scattering amplitude belongs to. After having the function space specified we should need mathematical expansion theorems. Our present knowledge of strong interactions is not sufficient for specifying this function space precisely. However, we do know that the scattering amplitude does not belong to those function spaces for which the above-mentioned expansion theorems are known. (These problems are investigated in details in ref. 4.) One can say only that generally both unitary and non-unitary Poincaré and $SL(2,C)$ representations appear in the expansions (3.1.13) and (3.1.15) . It follows, that we must take the phenomenological standpoint that we consider these expansions useful only if a (possibly infinite, c.f. daughters) number of representations exists which produces a good approximation of the experimentally measured scattering amplitude (see, e.g. the $s^{\alpha(t)}$ behaviour), and, at the same time, it fulfills some fundamental theoretical requirements (e.g. analyticity) .

Our next comment concerns the function $T_{\Sigma,\lambda;\Sigma',\lambda'}(t,W^{(+)})$ which stands in (3.1.13) and (3.1.15) for the reduced matrix element of

$$\langle t,W^{(+)};j_o,\sigma,j,m;\Sigma,\lambda\ |T|\ t',W^{(+)\prime};j_o',\sigma',j',m';\Sigma',\lambda'\rangle \qquad (3.2.1)$$

Owing to the Wigner-Eckart theorem the reduced matrix element does not depend on the quantum numbers j_o , σ , j , m , and is diagonal in the variables t and $W^{(+)}$. Moreover, it is nothing else but the old, well-known partial wave amplitude. We want to emphasize that one need not to suspect any complications when evaluating the matrix element (3.2.1) because diagonality in the four-momentum is not explicit. Momentum conservation is a part of the statement that T is a Poincaré invariant operator, and we make use of it when applying the Wigner-Eckart theorem. Specially momentum conservation is realized in the form of the functions which give the angles of the (+) and (-) -type Lorentz transformations in terms of the masses and the invariant variables s and t. Nevertheless, it is a sensible question how momentum conservation can be seen when dealing with matrix elements like the one in (3.2.1), but it is not easier to answer this question than to demonstrate the diagonality in (in fact more, the independence on) the j_0, σ quantum numbers for the reduced matrix elements when taking momentum basis instead of $SL(2,C)$ basis for the Poincaré representations.

Finally we discuss the outlooks of the expansions given in the previous paragraph. First of all we mention that from both forms we can obtain the old partial wave expansion or the Regge pole expansion, depending on the values of the variables s, t, if we sum up in all the indices but for $W^{(+)}$. These summations can be performed by making use only of completeness relations. We must notice, that our formalism involves an approach to the problem of the $t=0$ point, which is different from the usual one. After performing the above-mentioned summations we obtain the little-group expansion of the scattering amplitude:

$$f^t_{\Sigma,\lambda;\Sigma^*\lambda^*}(s,t) = \sum_W \delta_{jw}\, T_{\Sigma,\lambda;\Sigma^*\lambda^*}(t,W)\, D^j_{\lambda\lambda^*}(R). \qquad (3.2.2)$$

Usually, people say that when going to $t=0$ those $D^j_{\lambda\lambda'}(R)$ functions which correspond to the Regge poles become singular, and an infinite number of them is necessary to eliminate the singularity and to produce the $s^{\alpha(t)}$ behaviour. Instead, we say, that for the non-unitary representations j tends to infinity when t goes to zero, and the $d^j_{mm'}(\theta)$ part of the $D^j_{\lambda\lambda'}(R)$ function becomes a Bessel-function $J_{m'-m}(\rho\theta)$ (c.f. Appendix A). That is, the expansion (3.2.2) will again be an expansion with respect to the representations of the actual little-group, the $E(2)$ group. In this case no singularity appears, but the contribution of one term containing a Bessel-function does not produce the $s^{\alpha(t)}$ behaviour. In fact, we must again suppose the presence of an infinite number of such terms to produce the $s^{\alpha(t)}$ behaviour.

Of course, this expansion is not useful. We look for expansions in which one or two terms give a good approximation of the scattering amplitude even at $t = 0$. The solution presents itself: let us perform first the summation over the little-group representations (i.e. the Regge poles) instead of j_0 , σ . In this way we obtain an expansion of the scattering amplitude with respect to Lorentz group representations, and it is clear from the detailed study of the two-particle kinematics that no analyticity problems appear. This is, in our opinion, the clearest group-theoretical formulation of the generally accepted fact, that Lorentz poles are created by the "conspiracy" of Regge poles. Moreover, this formalism seems to be the most natural generalization of the expansions of the scattering amplitude in the equal-mass case with respect to $O(4)$ representations in the unphysical region and with respect to $SL(2,C)$ representation functions in the crossed channel, given by Freedman-Wang and Toller, respectively. The explicit form of the expansions obtained in the present paper is unfortunately complicated, and we must postpone their further discussion to forthcoming publications. We notice only one remarkable feature of our approach: The $t = 0$ point lost its significance from the point of view of the expansion with respect to Lorentz group representations. There is no wonder, because our approach does not follow the usual conception of the expansion in terms of little-group representations. But now we must reexamine the notion of "symmetry breaking" for $t \neq 0$.

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Appendix A.

1. The interpolating group

It was proved in the former paragraphs that it is extremely useful to work with other little-groups, that is, with other subgroups of the homogeneous Lorentz group than the usual rotation group, when we want to investigate some properties of the scattering amplitude. Now we give a detailed account of these "unusual" subgroups and the Lorentz transformations when decomposed with respect to these subgroups.

As it is well-known, the generators of any little-group can be found, if we take the operators

$$W_\mu = \frac{1}{2}\,\varepsilon_{\mu\nu\rho\kappa}\,M_{\nu\rho}\,P_\kappa \tag{A1}$$

over states having the special four-momentum which we want to concern ourselves with. Specially, we want p_μ to be the four-vector

$$p_\mu = p_o\left(1,0,0,v\right) , \tag{A2}$$

where $p_o > 0$ is fixed and $0 \le v < \infty$. After dividing by p_o we get three independent operators since $P_\mu W_\mu = 0$:

$$S_1(v) = M_1 + vN_2 , \qquad S_2(v) = M_2 - vN_1 , \qquad S_3(v) = M_3 . \tag{A3}$$

They can be used for generating the little group. Their commutation relations read:

$$[S_1, S_2] = i\left(1-v^2\right)S_3 , \qquad [S_1, S_3] = -iS_2 , \qquad [S_2, S_3] = iS_1 . \tag{A4}$$

It is obvious from eqs. (A3) and (A4) that the S_i operators form subalgebra of the Lorentz algebra. At the points $v = 0$, 1 and $\sqrt{2}$ we have the well-known $SU(2)$, $E(2)$ and $SU(1,1)$ algebras, respectively, and changing the values of v the little groups turn smoothly into one another. For this reason, we shall call the group generated by the S_i operators interpolating group (IG).

The general form for the group elements is:

$$G = \exp\left[-i\sum_{i=1}^{3}\alpha_i\,S_i(v)\right] .$$

The range of the parameters will be discussed later. It is easy to show that the group can be parametrized in the Eulerian way, as it is usual in the special cases $v = 0, 1, \sqrt{2}$:

$$G^{-i\varphi S_3(v)} e^{-i\theta S_2(v)} e^{-i\varphi' S_3(v)} . \tag{A5}$$

For example, for $v < 1$ we get after simple calculation, that

$$e^{-i\Sigma\alpha_i S_i(v)} = e^{-itS_3(v)} e^{-iuS_2(v)} e^{-i\gamma S_3(v)} e^{iuS_2(v)} e^{itS_3(v)} , \tag{A6}$$

where

$$t = \text{arc tg} \frac{\alpha_2}{\alpha_1} ,$$

$$u = \frac{1}{\sqrt{1-v^2}} \text{arc tg} \frac{\sqrt{(1-v^2)(\alpha_1^2+\alpha_2^2)}}{\alpha_3} , \tag{A7}$$

$$\gamma = \sqrt{(1-v^2)(\alpha_1^2 + \alpha_2^2) + \alpha_3^2} .$$

Now the composition rule is necessary for writing (A6) into the compact form (A5) . It will be clear later that this rule is the same as the one for the rotation group. In the case $v \geq 1$ the calculation can be carried out similarly.

2. The representations of the $S_i(v)$ algebra

Before examining the representations of the IG, it will be useful to discuss the Hermitian representations of the Lie algebra of the generators $S_i(v)$. We shall make use of them for obtaining the unitary irreducible representations of the IG.

First we assume that $0 \leq v < 1$ and take the familiar Lie algebra of $SU(2)$:

$$[J_i, J_j] = i\, \varepsilon_{ijk} J_k . \tag{A8}$$

Next, we subject the operators J_i to the transformation:

$$J_1^\lambda = \lambda J_1 , \quad J_2^\lambda = \lambda J_2 , \quad J_3^\lambda = J_3 . \tag{A9}$$

Here we used the notation $\lambda = \sqrt{1-v^2}$. The algebra of the operators J_i^λ is the same as that of the $S_i(v)$'s . Since the transformation (A9) is real and nonsingular except the point $\lambda = 0$, we conclude that the algebra $(A4)$ has representations of the same kind for $0 < v < 1$, as it has at $v = 0$. Namely, all the hermitian irreducible representations are finite dimensional. The linear space on which the representation is based can be spanned by the eigenstates of the operator $S_3(v)$:

$$S_3(v)\,|v;j,m\rangle = m\,|v;j,m\rangle \quad , \tag{A10}$$

where m is integer or half-integer. The different irreducible representations have different maximal weight $j = \max m$. We write also v as index for the basis vectors to denote the actual value of v for which we want to represent the algebra $(A4)$. The eigenvalue of the Casimir operator $S_1^2(v) + S_2^2(v) + (1-v^2)\,S_3^2(v)$ is a characteristic quantity for the irreducible representations:

$$[S_1^2 + S_2^2 + (1-v^2)S_3^2]\;|v;j,m\rangle = (1-v^2)\;j\;(j+1)\,|v;j,m\rangle \quad . \tag{A11}$$

Writing like this the eigenvalue of the Casimir operator, we lay stress on the connection of the $S_i(v)$ algebra with the one at $v = 0$. It is obvious from the commutation relations that the operators $S_\pm(v) = S_1(v) \pm iS_2(v)$ are the raising and lowering ones. The matrix form of the generators in the $|v;\, j,\, m\rangle$ basis is:

$$\langle v;j,m|S_1|v;j,m'\rangle = \frac{\sqrt{1-v^2}}{2}\left[\sqrt{j(j+1)-m(m-1)}\;\delta_{m'm-1} + \sqrt{j(j+1)-m(m+1)}\;\delta_{m'm+1}\right]$$

$$\langle v;j,m|S_2|v;j,m'\rangle = -i\,\frac{\sqrt{1-v^2}}{2}\left[\sqrt{j(j+1)-m(m-1)}\;\delta_{m'm-1} - \sqrt{j(j+1)-m(m+1)}\;\delta_{m'm+1}\right]$$

$$\langle v;j,m|S_3|v;j,m'\rangle = m\;\delta_{mm'} \quad . \tag{A12}$$

Now we turn to the case $v = 1$. It is highly an exceptional point, because then the algebra $(A4)$ is the not semi-simple $E(2)$ algebra. This break in the structure of the Lie-algebra is strongly correlated with the fact that the transformation $(A9)$ turns to be singular at $v = 1$. What actually happens is that when changing continuously the value of v from 0 to 1 the $SU(2)$ algebra becomes deformed into the $E(2)$ algebra. This phenomenon is called contraction, and one has to apply special methods for obtaining faithful representations of $E(2)$ from the representation $(A12)$. A standard method is to choose infinitely rising sequence of j values when going to $v = 1$. Namely, taking

$$j = \left[\frac{\rho}{\sqrt{1-v^2}}\right] \quad \text{or} \quad j = \left[\frac{\rho}{\sqrt{1-v^2}}\right] + \frac{1}{2} \tag{A13}$$

where ρ is a positiv number and $[c]$ denotes the integer part of the number c. , we get the matrices:

$$\lim_{\substack{v\to 1\\ j\to\infty}} \langle v;j,m|S_1|v;j,m'\rangle = \frac{1}{2}\,\rho\left(\delta_{m'm+1} + \delta_{m'm-1}\right),$$

$$\lim_{\substack{v\to 1\\ j\to\infty}} \langle v;j,m|S_2|v;j,m'\rangle = -\frac{i}{2}\,\rho\left(-\delta_{m'm+1} + \delta_{m'm-1}\right), \tag{A14}$$

$$\lim_{\substack{v\to 1\\ j\to\infty}} \langle v;j,m|S_3|v;j,m'\rangle = m\,\delta_{mm'} .$$

These matrices are Hermitian and commute like the elements of the $E(2)$ Lie algebra. Consequently, we reached the result: the $S_i(1)$ algebra is represented by Hermitian operators over a linear space spanned by $|\rho,m\rangle$ basis vectors. For every value of ρ we have two kinds of representations depending on whether the eigenvalues of $S_3(1)$ are integer of half-integer. These representations are irreducible for any positive number ρ. The Casimir operator $S_1^2(1) + S_2^2(1)$ has the eigenvalue ρ^2 when acting on the basis vectors. An alternative definition of the eigenvalues of the Casimir operator instead of (A11) is:

$$\left[S_1^2(v) + S_2^2(v) + \left(1-v^2\right) S_3^2(v)\right] |v;k,m\rangle = \left[k^2 - \frac{1}{4}\left(1-v^2\right)\right]|v;k,m\rangle . \tag{A15}$$

We changed here the index j in the basis vectors, too. The connection between j and k can be written as follows:

$$j = -\frac{1}{2} + \frac{1}{\sqrt{1-v^2}}\,k . \tag{A16}$$

The index k has its advantage in remaining finite when $v \to 1$, not like j does.

We do not repeat the discussion for the case $v > 1$, we write only the two main properties of the representations:

$$S_3(v)\,|v;j,m\rangle = m\,|v;j,m\rangle \quad ,$$

$$\left[-S_1^2(v) - S_2^2(v) + (v^2-1)\,S_3^2(v)\right]|v;j,m\rangle = (v^2-1)\,j(j+1)\,|v;j,m\rangle \; . \tag{A17}$$

The possible values of j and m , that is, the types of the $SU(1,1)$ representations, are well-known [15]. Attention must be paid to the point $v = 1$. Here we get the $E(2)$ algebra as the contraction of the $SU(1,1)$ algebra. Our method for getting faithful representations is the same as in the previous case. It is noteworthy, that we cannot reach the same representation of $E(2)$ using different kinds of $SU(1,1)$ representations at the limiting procedure. Namely, we can get the principal series of $E(2)$ using that one of $SU(1,1)$, but we cannot get any representation of $E(2)$ from the discrete series of $SU(1,1)$. Writing again not j but $j=-\frac{1}{2}+\frac{1}{\sqrt{1-v^2}}k$ with a continuous real parameter k, we see k is the most convenient parameter for distinguishing the irreducible representations.

3. Thel unitary representations of the IG

We have worked with the algebra rather than the finite group elements. Now we apply the results for obtaining the unitary representations of the one-parameter group elements

$$U(\Theta;v) = \exp\left[-i\Theta S_2(v)\right] .$$

It is certainly true that the unitary representations of the IG can be based on the linear spaces defined in the previous part. To find the representation matrix elements we follow the standard method, described e.g. in ref. 12. That is, we seek the solution, regular at $\Theta = 0$, of the differential equation

$$\left[\frac{d^2}{d\Theta^2} + \sqrt{1-v^2}\,\mathrm{ctg}\Theta\sqrt{1-v^2}\,\frac{d}{d\Theta} - \frac{1-v^2}{\sin^2\Theta\sqrt{1-v^2}}\left(m^2 + m'^2 - 2mm'\cos\Theta\sqrt{1-v^2}\right) - \right.$$

$$\left. - (1-v^2)\left(\frac{1}{4} - \frac{k^2}{1-v^2}\right)\right] d^{kv}_{mm'}(\Theta) = 0 \quad . \tag{A18}$$

Henceforth we use the notation for any x quantity:

$\bar{x} = x\sqrt{1-v^2} = x\alpha = \frac{x}{\kappa}$. In eq. (A18) $d^{kv}_{mm'}$ stands for the matrix element

$$\langle v;k,m|e^{-i\Theta S_2(v)}|v;k,m'\rangle .$$

Equation (A18) can be solved easily by displacing its singularities to $0, 1, \infty$, when it casts hypergeometric form in the variable $z = \cos\bar{\Theta}$. The solution for the case $m \le m'$ can be written as (in the case $m>m'$ one must simply change m and m'):

$$d^{kv}_{mm'}(\Theta) = N(k,v;m,m')\left(\frac{1+\cos\bar{\Theta}}{2}\right)^{\frac{1}{2}(m+m')}\left(\frac{1-\cos\bar{\Theta}}{2}\right)^{\frac{1}{2}(m-m')} \times \tag{A19}$$

$$\times F\left(\frac{1}{2}+m'-k\kappa;\ \frac{1}{2}+m'+k\kappa;\ m'-m+1;\ \frac{1-\cos\bar{\Theta}}{2}\right).$$

Here $N(k,v;m,m')$ is a normalization factor defined to be one for $m = m'$. Obviously, (A19) gives the well-known $SU(2)$ functions for $v = 0$ and the $SU(1,1)$ functions for $v = \sqrt{2}$ except for that our notation is $j=\frac{1}{2}+k$ and $j = \frac{1}{2} + ik$, respectively. In the limit $v = 1$ it can be written:

$$d^{kv}_{mm'}(\Theta)\xrightarrow[k\to\rho]{v\to1} N(\rho,1;m,m')\ \frac{\bar{\Theta}^{m'-m}}{2}\ F\left(\frac{k}{\lambda},\ \frac{-k}{\lambda}\ ;\ m'-m+1;\ \frac{\Theta}{4}\right), \tag{A20}$$

and, from Hansen's formula [22]

$$\lim_{a,b\to\infty} x^{\frac{1}{2}(c-1)}\ F\left(a,\ -b;\ c;\ \frac{x}{ab}\right) = \Gamma(c)\ J_{c-1}\left(2\sqrt{x}\right),$$

and we obtain that

$$\lim_{\substack{v\to1\\ k\to\rho}} d^{kv}_{mm'}(\Theta) = d^{\rho}_{mm'}(\Theta) \sim J_{m'-m}(\rho\Theta)\ .$$

Here $J_n(x)$ denotes the Bessel function of the first kind. This result means that the solution (A19) gives the $E(2)$ representation functions as well. The normalization factor $N(k,v;m,m')$ is determined from (A12):

$$N(k,v;m,m') = \frac{1}{\Gamma(m'-m+1)}\left[\frac{\Gamma\left(\frac{1}{2}+m'+\frac{k}{\lambda}\right)\ \Gamma\left(\frac{1}{2}-m+\frac{k}{\lambda}\right)}{\Gamma\left(\frac{1}{2}+m+\frac{k}{\lambda}\right)\ \Gamma\left(\frac{1}{2}-m'+\frac{k}{\lambda}\right)}\right]^{1/2}.$$

We have not spoken yet about the ranges of the group parameters. Remembering to the procedure which gave connection between the representations of the $S_i(v)$ generators for different values of v, it is evident that

$$\text{a)}\quad \text{if}\quad 0 \le v < 1 \qquad -\pi\kappa \le \alpha_i \le \pi\kappa \;; \qquad -\pi \le \alpha_3 \le \pi$$

$$i = 1,2$$

$$\text{b)}\quad \text{if}\quad 1 \le v \qquad -\infty < \alpha_i < \infty \;; \qquad -\pi \le \alpha_3 \le \pi \,.$$

In terms of Euler parameters:

$$\text{a)}\quad \text{if}\quad 0 \le v < 1 \qquad 0 \le \theta \le \pi\kappa \qquad 0 \le \phi,\phi' < 2\pi$$

$$\text{b)}\quad \text{if}\quad 1 \le v \qquad 0 \le \theta < \infty \qquad 0 \le \phi,\phi' < 2\pi \,.$$

The properties of the $SU(2)$, $SU(1,1)$ and $E(2)$ functions are well-known. Since the functions, given by (A19), are in a very simple connection with these special cases, it can be justified even directly, that the whole group is covered when taking the parameters from the ranges specified, and it is covered only once.

The invariant measure for integration over the IG is:

$$d\mu_v = -\frac{1}{8\pi^2}\,\kappa\,\sin\bar{\theta}\,d\theta\,d\phi\,d\phi' \,.$$

The normalization is:

$$\int_{G(v)} d\mu_v = \begin{cases} \kappa^2 & \text{for} \quad 0 \le v \le 1 \\ \infty & \text{for} \quad v \ge 1 \,. \end{cases}$$

This choice gives the usual measures in the $SU(2)$, $SU(1,1)$, $E(2)$ cases. The orthogonality relations are:

$$\int D^{jv}_{mm'}(\phi,\theta,\phi')\, D^{j'v}_{m_1m_1'}(\phi,\theta,\phi')\, d\mu_v = \kappa^2\,\frac{\delta_{jj'}}{2j+1}\,\delta_{mm_1}\,\delta_{m'm_1'} \,.$$

if $0 \le v < 1$, and

$$\int D^{jv}_{mm'}(\phi,\theta,\phi')\, D^{j'v}_{m_1m_1'}(\phi,\theta,\phi')\, d\mu_v = \kappa^2\,\frac{\delta(ij-ij')}{i(2j+1)}\,\delta_{mm_1}\,\delta_{m'm_1'}$$

if $v \ge 1$ and we deal with the principal series.

4. <u>The IG as subgroup of the Lorentz group. The representations on four-vectors</u>

Next we turn our attention to the Lorentz group. As it is known, its Lie algebra is spanned by six generators, M_i, N_i, $i = 1, 2, 3$, commuting as

$$[M_i, N_j] = i\varepsilon_{ijk} N_k \;,$$
$$[M_i, M_j] = -[N_i, N_j] = i\varepsilon_{ijk} M_k \;. \tag{A21}$$

If we introduce the linear combinations

$$S_1(v) = M_1 + vN_2 \;, \qquad S_2(v) = M_2 - vN_1 \;, \qquad S_3 = M_3 \tag{A22}$$

we get exactly the same Lie algebra as we examined previously. With the help of eq. (A22) we can write an element of the IG in the 4x4 representation:

$$S(\alpha,\beta,\gamma) \equiv e^{-i\alpha S_3} e^{-i\beta S_2} e^{-i\gamma S_3} = \tag{A23}$$

$$\begin{bmatrix} \kappa^2(1-v^2\cos\bar\beta) & \kappa v\sin\bar\beta\cos\gamma & -\kappa v\sin\bar\beta\sin\gamma & -\kappa^2 v(1-\cos\bar\beta) \\ \kappa v\cos\alpha\sin\bar\beta & \cos\alpha\cos\bar\beta\cos\gamma-\sin\alpha\sin\gamma & -\cos\alpha\cos\bar\beta\sin\gamma-\sin\alpha\sin\gamma & -\kappa\cos\alpha\sin\bar\beta \\ \kappa v\sin\alpha\sin\bar\beta & \sin\alpha\cos\bar\beta\cos\gamma-\cos\alpha\sin\gamma & -\sin\alpha\cos\bar\beta\sin\gamma+\cos\alpha\cos\gamma & -\kappa\sin\alpha\sin\bar\beta \\ \kappa^2 v(1-\cos\bar\beta) & \kappa\sin\bar\beta\cos\gamma & -\kappa\sin\bar\beta\sin\gamma & \kappa^2(\cos\bar\beta-v^2) \end{bmatrix}.$$

The elements of this matrix are analytic at $v=1$:

$$\lim_{v\to1} e^{-i\beta S_2} = \begin{bmatrix} 1+\frac{\beta^2}{2} & -\beta & 0 & -\frac{\beta^2}{2} \\ \beta & 1 & 0 & -\beta \\ 0 & 0 & 1 & 0 \\ \frac{\beta^2}{2} & \beta & 0 & 1-\frac{\beta^2}{2} \end{bmatrix}. \tag{A23a}$$

In the previous part we have learned a method for obtaining faithful representations after contraction. The representations were unitary there, and their dimension altered with v. Here another method is exhibited: we do not make the dimension changed, but we always take non-unitary representations.

It can be immediately seen that the matrices (A23) transform the vector $(1, 0, 0, v)$ into the zero vector $(0, 0, 0, 0)$ as expected.

The elements of the Lorentz group are usually given in the Euler parametrized form:

$$\Lambda = e^{-i\alpha M_3} e^{-i\beta M_2} e^{-i\gamma N_1} e^{-i\delta M_3} e^{-i\varepsilon M_2} e^{-i\phi M_3} . \tag{A24}$$

(N_2 or N_3 are as good as N_1 in this formula. In some cases we prefer N_3, in others N_1). The question arises whether any other little group can be used for Euler parametrization instead of the rotation group. The answer is affirmative. To see it one must take the normal form

$$\Lambda = e^{-i\alpha_i M_i + i\beta_i N_i} = e^{-i\gamma_k G_k} , \quad \begin{array}{l} i = 1,2,3, \\ k = 1,\dots 6, \end{array}$$

where $G_i = M_i$ and $G_{i+3} = N_i$. In the vector space of the operators we perform a non-singular transformation $G' = UG$, where

$$U = \left[\begin{array}{ccc|ccc} 1 & & & v & & \\ & 1 & & & -v & \\ & & 1 & & & \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right]$$

Comparing with eq. (A22) : $G_i' = S_i$, $G_{i+3}' = N_i$. As U is regular for any v, there is a one-to-one correspondence between the elements $\exp(-i\gamma_k G_k)$ and $\exp(-i\gamma_k' G_k')$. The method for finding the Euler parametrized form is similar in both cases. As a consequence we shall write the elements of the ortochronous Lorentz group in the form:

$$\Lambda = e^{-i\varphi S_3} e^{-i\vartheta S_2} e^{-i\xi N_1} e^{-i\alpha S_3} e^{-i\beta S_2} e^{-i\gamma S_3}$$

$$0 \le \varphi, \alpha, \gamma < 2\pi , \quad 0 \le \xi < \infty , \quad 0 \le \vartheta, \beta \le \frac{\pi}{1-v^2} \quad \text{if} \quad v < 1$$

$$0 \le \vartheta, \beta < \infty \quad \text{if} \quad v \ge 1 . \tag{A25}$$

It will not be useless to classify the orbits for Λ. (Orbit is the set of the points $\Lambda \overset{o}{q}$, where $\overset{o}{q}$ is a fixed four-vector and the parameters of Λ run over all the possible values.) To achieve the classification we must solve the equation:

$$\Lambda_{\mu\nu}(\phi',\Theta,\xi,\alpha,\beta,\gamma)\, \overset{o}{q}_\nu = p_\mu$$

where $p_\mu = (p_o, p\sin\omega\cos\phi, p\sin\omega\sin\omega, p\cos\omega)$ is an arbitrary four-vector. According to the value of p_μ^2 there are three cases:

$$p_\mu^2 > 0 \qquad p_o = \sqrt{p_\mu^2}\cosh\alpha' \qquad p = \sqrt{p_\mu^2}\sinh\alpha'$$

$$p_\mu^2 = 0 \qquad p_o = p = e^{\alpha'} \tag{A26}$$

$$p_\mu^2 < 0 \qquad p_o = \sqrt{-p_\mu^2}\sinh\alpha' \qquad p = \sqrt{-p_\mu^2}\cosh\alpha' .$$

We choose $\overset{o}{q}_\nu$ to be $q_o(1,0,0,v)$. Then the parameters α, β, γ, are irrelevant. Working with the four-dimensional representation we get:

$$\phi = \phi" , \quad \cosh\xi = \frac{p_o - v\, p\cos\omega}{q_o} + v^2$$

$$\frac{q_o}{p}\kappa v\, \{\cosh\xi(1-\cos\bar{\Theta}) + v^2\cos\bar{\Theta} - 1\} = \sin\omega - \lambda \mathrm{ctg}\bar{\Theta}\cos\omega . \tag{A27}$$

If $v < 1\ (p_\mu^2 > 0)$, there is no problem with eq. (A27). For $v > 1\ (p_\mu^2 < 0)$ we have to allow both $q_o > 0$ and $q_o < 0$ for covering the whole orbit. In a similar way we can find the Λ_v transformation which connects two fixed four-vectors $q_1 = m_1(\cosh\eta,0,0,\sinh\eta)$ and $q_2 = m_2(\cosh\eta,0,0,-\sinh\eta)$ with any pair p_1, p_2 satisfying $(q_1+q_2)^2 = (p_1+p_2)^2$. The notation is: $v = \frac{(q_1+q_2)_z}{(q_1+q_2)_o}$. That is, we are looking for transformation with the property: $p_1 = \Lambda_v q_1$, $p_2 = \Lambda_v q_2$. We only sketch the way of the calculation. First, from the equation $p_1+p_2 = \Lambda_v(q_1+q_2)$ we get ϕ', Θ, ξ similarly to eq. (A27). Then

$$\Lambda_v^{-1}(\phi,\Theta,\xi)(p_1 - p_2) = S(\alpha,\beta,\gamma)\,(q_1 - q_2) \tag{A27a}$$

gives α and β. The angle γ remains unconstrained.

5. <u>The function</u> $\langle j_o, \sigma, k, m; v | j_o, \sigma, k', m'; v' \rangle$.

As is well-known, four quantum numbers are necessary for labeling the irreducible representations of the Lorentz group. Besides the eigenvalues of the Casimirians $\underline{M}^2 - \underline{N}^2$ and $\underline{MN}$ we can choose those of the Casimir operator of an IG and of M_3 . It is natural to ask for the transformation coefficients between two different basis systems the vectors of which are $|j_o, \sigma, k, m; v\rangle$ and $|j_o, \sigma, k', m'; v'\rangle$.

We apply the method described by Delbourgo et al. in ref. 21. The basic equation is:

$$e^{-i\xi N_3}\, e^{-i\chi_A J_A} = e^{-i\chi_B J_B}\, e^{-i\eta(N_1 - M_2)}\, e^{-i\alpha N_3} . \tag{A28}$$

Here J_A and J_B stand for the $S_2(v)$ generators with v_A , and v_B , respectively. In some cases χ_A takes all its possible values, while χ_B covers only a part of its domain of definition or vice versa. In these cases an additional factor $\exp(-i\pi M_2)$ is needed. In the 2x2 representation eq. (A28) reads as

$$\begin{bmatrix} e^{\frac{\xi}{2}} & 0 \\ 0 & e^{-\frac{\xi}{2}} \end{bmatrix} \begin{bmatrix} \cos\frac{\bar{\chi}_A}{2} & -\frac{1-v_A}{1+v_B}\sin\frac{\bar{\chi}_A}{2} \\ \frac{1+v_A}{1-v_A}\sin\frac{\bar{\chi}_A}{2} & \cos\frac{\bar{\chi}_A}{2} \end{bmatrix} = \begin{bmatrix} \cos\frac{\bar{\chi}_B}{2} & -\frac{1-v_B}{1+v_B}\sin\frac{\bar{\chi}_B}{2} \\ \frac{1+v_B}{1-v_B}\sin\frac{\bar{\chi}_B}{2} & \cos\frac{\bar{\chi}_B}{2} \end{bmatrix} \begin{bmatrix} e^{\frac{\alpha}{2}} & \eta e^{\frac{\alpha}{2}} \\ 0 & e^{-\frac{\alpha}{2}} \end{bmatrix} \tag{A29}$$

Hence

$$\frac{1+v_A}{1-v_A}\, e^{-\xi}\, \mathrm{tg}\,\frac{\bar{\chi}_A}{2} = \frac{1+v_B}{1-v_B}\, \mathrm{tg}\,\frac{\bar{\chi}_B}{2} ,$$

$$\frac{1+v_A}{1-v_A}\sin\bar{\chi}_A = e^{\alpha}\,\frac{1+v_B}{1-v_B}\sin\bar{\chi}_B , \tag{A30}$$

$$\frac{1-v_B}{1+v_B}\, e^{-\frac{\xi}{2}} \cos\frac{\bar{\chi}_A}{2} \sin\frac{\bar{\chi}_B}{2} - \frac{1-v_A}{1+v_A}\, e^{\frac{\xi}{2}} \sin\frac{\bar{\chi}_A}{2} \cos\frac{\bar{\chi}_B}{2} =$$

$$= e^{\alpha}\eta\, \frac{1}{1+v_B}\left(1 + v_B \cos\bar{\chi}_B\right) .$$

As it can be seen, the afore-mentioned problem arises in eq. (A30) if either v_A or v_B or both are bigger than 1 . After adding a factor $\exp\left(-i\pi M_2\right)$ to the right-hand-side of the first two equations of (A30) we get:

$$-\frac{1+v_A}{1-v_A} e^{-\xi} \operatorname{ctg} \frac{\chi_A}{2} = \frac{1+v_B}{1-v_B} \operatorname{tg}\bar{\chi}_B ,$$

$$-\frac{1-v_A}{1+v_A} \sin\bar{\chi}_A = e^{\alpha} \frac{1+v_B}{1-v_B} \sin\bar{\chi}_B . \qquad (A31)$$

With the same trick as in ref. 21 we obtain:

$$d^{j_o\sigma}_{k_v m k'_{v'}}(\xi) = N \int d\chi_B \, d^{kv}_{mj_o}\left(\chi_B\right) d^{k'v'}_{mj_o}\left(\chi_A\right) e^{\alpha(\sigma-1)} . \qquad (A32)$$

We do not go into more details, they can be extracted from ref. 21. We note, that in the case when v_A or/and v_B are bigger than 1 , $d^{j_o\sigma}$ splits into two irreducible representations, corresponding to eq. (A30) or eq. (A31) .

Appendix B.

Generalized partial wave analysis

In this section we compute the following bracket: $\langle s,W;j_o,\sigma,k,m;\Sigma,\lambda|p_1, s_1, \lambda_1 ; p_2, s_2, \lambda_2\rangle$. We do it in several steps, defining successively the following functions (with $P=p_1+p_2$, $Q=p_1-p_2$) :

$$\langle s,W;\underline{P},W_3,\lambda_1,\lambda_2|P,Q,\lambda_1,\lambda_2\rangle , \quad \langle s,W;\underline{P},W_3,\Sigma,\lambda|s,W;\underline{P},W_3;\lambda_1,\lambda_2\rangle ,$$

$$\langle s,W;j_o,\sigma,k,m;\Sigma,\lambda|s,W;\underline{P},W_3;\Sigma,\lambda\rangle .$$

a) $\langle s,W;\underline{P},W_3;\lambda_1,\lambda_2|P,Q,\lambda_1,\lambda_2\rangle$.

This matrix element is the one, which appears in the partial wave analysis (PWA) with respect to the little-group of P . W and W_3 are eigenvalues of the little-group Casimirian and a diagonalized generator, respectively. The PWA is elaborated in detail e.g. in ref. 17. for SU(2) and in ref. 11. for all the other cases, for E(2) also in ref. 20.

All the $|P, Q, \lambda_1 \lambda_2>$ vectors can be obtained from a standard one $|P, \overset{o}{Q}, \lambda_1, \lambda_2>$ with the help of a little-group transformation R_W

$$|P, Q, \lambda_1, \lambda_2> = R_W |P, \overset{o}{Q}, \lambda_1, \lambda_2> . \tag{B1}$$

The representations of the little-group form a complete system:

$$\sum_{W,\alpha,\beta} D^W_{\alpha\beta}(\phi,\theta,\psi)\, D^W_{\alpha\beta}(\phi',\theta',\psi') \sim \delta(\phi-\phi')\, \delta(\psi-\psi')\, \delta(\theta-\theta') , \tag{B2}$$

If the group is not compact, $\sum_W$ denotes both integration and summation over all the unitary representations. By making use of eq. (B2) the following operator equality can be proved:

$$R(\phi,\theta,\psi) = \sum_{W,\alpha,\beta} \int dG(\phi',\theta',\psi')\, D^W_{\alpha\beta}(\phi,\theta,\psi)\, D^W_{\alpha\beta}(\phi',\theta',\psi')\, R(\phi',\theta',\psi') \tag{B3}$$

where $dG(\phi, \theta, \psi)$ is the group measure. Let us define

$$|s,W;\underline{P},\alpha,\beta;\lambda_1,\lambda_2> = \int dG\, D^W_{\alpha\beta}(R'_W)\, R'_W |P,\overset{o}{Q};\lambda_1,\lambda_2 \; \frac{\sqrt{s}}{\Delta^{1/4}(s,m_1^2,m_2^2)} . \tag{B4}$$

We may choose the standard vector $\overset{o}{Q}$ in such a way, that $\overset{o}{Q}_x = \overset{o}{Q}_y = 0$. Then the integration over ψ' can be performed, and this gives a relation between β, λ_1, λ_2. So β is "superfluous" in the ket $|s,W;\underline{P},\alpha,\beta;\lambda_1,\lambda_2>$. This is the consequence of the fact that two angles are enough for fixing the direction of $\overset{o}{Q}$. In the following we omit β on the left-hand-side of eq. (B4). Inserting (B4) and (B3) into (B1) we obtain:

$$|P, Q; \lambda_1, \lambda_2> = \sum_{W,W_3} (2W+1) \frac{\sqrt{s}}{\Delta^{1/4}(s,m_1^2,m_2^2)} \delta_{\beta, f(\lambda_1\lambda_2)} |s,w;\underline{P},W_3;\lambda_1,\lambda_2>. \tag{B5}$$

Here the symbol $f(\lambda_1, \lambda_2)$ means either $\lambda_1 - \lambda_2$ or $\lambda_1 + \lambda_2$ depending on the group structure.

b) $<s,W;\underline{P},W_3;\lambda_1,\lambda_2|s,W;\underline{P},W_3;\Sigma,\lambda>$.

This expression was discussed in Sect. 2: here we repeat only the result:

$$\langle s,W;\underline{P},W_3,\lambda_1,\lambda_2|s,W;\underline{P},W_3;\Sigma,\lambda\rangle = \sum_{s'} A_{\Sigma s'}\, C^{s_1\lambda_1,s_2-\lambda_2}_{s'\lambda} . \tag{B6}$$

c) $\langle s,W;j_o,\sigma,k,m;\Sigma,\lambda|s,W;\underline{P},W_3;\Sigma,\lambda\rangle$.

The method is similar to the one used in a). All the $|P,Q,\lambda_1,\lambda_2\rangle$ states can be obtained from a given standard state $|\overset{o}{P},\overset{o}{Q},\lambda_1,\lambda_2\rangle$ by homogeneous Lorentz transformation, where $\overset{o}{P} = p_o(1,0,0,v)$ and $\overset{o}{Q}_x=\overset{o}{Q}_y=0$. Moreover, we have the transformation rule

$$\Lambda\,|\overset{o}{P},\overset{o}{Q};\lambda_1,\lambda_2\rangle = \sum_{\lambda_1'\lambda_2'} D^{s_1}_{\lambda_1'\lambda_1}\, D^{s_2}_{\lambda_2'\lambda_2}\,|\Lambda\overset{o}{P},\Lambda\overset{o}{Q};\lambda_1',\lambda_2'\rangle . \tag{B7}$$

The Λ Lorentz transformation is Euler parametrized in terms of the IG, that is, of the little group of the $\overset{o}{P}$ vector. The analogue of eq. (B3) is:

$$\Lambda' = \sum_{\substack{j_o\sigma km\\k'm'}} \int d\Lambda\, D^{j_o\sigma}_{km,k'm'}(\Lambda)\, D^{j_o\sigma}_{km,k'm'}(\Lambda')\,\Lambda . \tag{B8}$$

Applying the results of a) we can write:

$$|\overset{o}{P},\overset{o}{Q};\lambda_1,\lambda_2\rangle = \sum_w (2W+1)\,\frac{\sqrt{s}}{\Delta^{1/4}(s,m_1^2,m_2^2)}\,|s,w,\underline{P},W_3;\lambda_1,\lambda_2\rangle . \tag{B9}$$

Since the Casimir operators of the Lorentz group do not commute with W_3, it is necessary to perform diagonalization in eq. (B9) . Since

$$\int d\Lambda\, D^{j_o\sigma}_{k\mu k'\mu'}(\Lambda)\,\Lambda\,|s,w;\underline{\overset{o}{P}},W_3;\Sigma,\lambda\rangle = \delta_{k'w}\,\delta_{\mu'\lambda}\,|s,w;j_o,\sigma,k,\mu;\Sigma,\lambda\rangle , \tag{B10}$$

we can write finally:

$$|P,Q;\Sigma,\lambda\rangle = \sum_{\substack{j_o,\sigma,k,m\\ \lambda_1',\lambda_2',\lambda_1,\lambda_2}} D^{j_o\sigma}_{jm,w}(\Lambda)\, d^{s_1}_{\lambda_1'\lambda_1}\, d^{s_2}_{\lambda_2'\lambda_2}\,|s,w;j_o,\sigma,k,m;\Sigma,\lambda\rangle \times$$

$$\times \langle\Sigma,\lambda|\lambda_1,\lambda_2\rangle\,\langle\lambda_1',\lambda_2'|\Sigma,\lambda\rangle\,(j_o^2-\sigma^2)\,\frac{\sqrt{s}}{\Delta^{1/4}(s,m_1^2,m_2^2)} . \tag{B11}$$

Appendix C.

The evaluation of the transformation coefficient

$$\langle s, W^{(+)}; j_o, \sigma, j, m; \Sigma^{(+)}, \lambda^{(+)} | \tau, W^{(-)}; \ell_o, \rho, \ell, \mu; \Sigma^{(-)}, \lambda^{(-)} \rangle .$$

We shall use the shorthand notation $\langle (+)|(-)\rangle$ for the matrix element in question. We can write, that

$$\langle (+)|(-)\rangle = \int \frac{d^3p_1 d^3p_2}{p_{1o}\, p_{2o}} \sum_{\lambda_1 \lambda_2} \langle (+)|p_1, s_1, \lambda_1; p_2, s_2, \lambda_2\rangle \langle p_1, s_1, \lambda_1; p_2, s_2, \lambda_2|(-)\rangle$$

$$\delta\left(p^{(+)2}-s\right)\,\delta\left(p^{-(2)}-\tau\right) = \int \frac{d^3p_1 d^3p_2}{p_{1o}\, p_{2o}} \sum_{\lambda_1\lambda_2, \ell'_v, j'_v} \times$$

$$\langle S, W^{(+)}; j_o, \sigma, j, m; O; \Sigma^{(+)}, \lambda^{(+)} | s, w^{(+)}; j_o, \sigma, j'_v, m; v; \Sigma^{(+)}, \lambda^{(+)}\rangle \times \tag{C1}$$

$$\times \langle s, w^{(+)}; j_o, \sigma, j'_v, m; v; \Sigma^{(+)}, \lambda^{(+)} | p_1, s_1, \lambda_1; p_2, s_2, \lambda_2\rangle \times$$

$$\times \langle p_1, s_1, \lambda_1; p_2, s_2, \lambda_2 | \tau, w^{(-)}; \ell_o, \rho, \ell'_v, \mu; v; \Sigma^{(-)}, \lambda^{(-)}\rangle \times$$

$$\times \langle \tau, w^{(-)}; \ell_o, \rho, \ell'_v, \mu; v; \Sigma^{(-)}, \lambda^{(-)} | \tau, w^{(-)}; \ell_o, \rho, \ell, \mu; O; \Sigma^{(-)}, \lambda^{(-)}\rangle\, \delta\left(p^{(+)2}-s\right)\, \delta\left(p^{(-)2}-\tau\right) ,$$

where $P^{(+)}=p_1+p_2$, $P_o^{(-)}=p_{1o}-p_{2o}$, $\underline{P}^{(-)} = \underline{p}_1+\underline{p}_2 \cdot j_v$ and ℓ_v are the eigenvalues of the Casimir operators of the little-group of $P^{(+)}$ and $P^{(-)}$, respectively. The quantity $\langle j|j_v\rangle$ and $\langle \ell|\ell_v\rangle$ is known from (A32). Inserting eq. (B11) into eq. (C1) we obtain:

$$\langle (+)|(-)\rangle = \frac{1}{2} \frac{\sqrt{s\tau}}{\Delta^{1/4}(s)\, \Delta^{1/4}(\tau)} \int d\Lambda^{(-)} \sum_{j_v, \ell_v} \left(j_o^2-\sigma^2\right)\left(\ell_o^2-\rho^2\right) \delta\left(p^{(+)2}-s\right)\, \delta\left(p^{(-)2}-\tau\right) \times$$

$$\times \langle j|j_v\rangle\, D^{j_o\sigma}_{j_v m w^{(+)}\lambda^{(+)}}\left(\Lambda^{(+)}\right)\, D^{\ell_o\rho}_{\ell_v \mu w^{(-)}\lambda^{(-)}}\left(\Lambda^{-}\right) \langle \ell_v|\ell\rangle \times$$

$$\times \sum_{\lambda'_i \lambda''_i} \langle \Sigma^{(+)}, \lambda^{(+)} | \lambda'_1, \lambda'_2\rangle\, d^{s_1}_{\lambda'_1\lambda''_1}(\Theta_1)\, d^{s_2}_{\lambda'_2\lambda''_2}(\Theta_2)\, \langle \lambda''_1, \lambda''_2 | \Sigma^{(-)}, \lambda^{(-)}\rangle \; . \tag{C2}$$

Here the parameters of $\Lambda^{(+)}$ and Θ_1 , Θ_2 depend on the parameters of Λ^- . We need eq. (C2) only at $\tau = (m_1 + m_2)^2$. The $\Delta(\tau)$ factor in the nominator seems to give a zero here, but a similar factor in the scattering amplitude (it appears in $\langle m_1, s_1, -m_2, s_2; \Sigma^{(-)}, \lambda^{(-)} | \tau, W^{(-)}; \ell_o, \rho, \ell, \mu; \Sigma^{(-)}, \lambda^{(-)} \rangle$) just cancels it out, so we need not bother ourselves because of that.

To find out the explicit dependence between the parameters of $\Lambda^{(+)}$ and $\Lambda^{(-)}$, we use eqs. (A27) and (A27a) with

$$p_1 = \Lambda^{(-)}(m_1, \underline{0}) \quad \text{and} \quad p_2 = \Lambda^{(-)}(-m_2, \underline{0}) \; .$$

The η parameter of the fixed $q_{1,2}$ vectors is given by

$$s = m_1^2 + m_2^2 - 2m_1 m_2 \cosh 2\eta \; .$$

To get Θ_1 and Θ_2, we must write the transformation

$$\Lambda^{(+)} = e^{-i\alpha S_3} e^{-i\beta S_2} e^{-i\gamma S_3} e^{-i\xi N_1} e^{-i\Theta S_2} e^{-i\phi S_3}$$

in the form:

$$\Lambda^{(+)} = e^{-i\alpha_1 M_3} e^{-i\alpha_2 M_2} e^{-i\alpha_3 M_3} e^{-i\alpha_4 N_3} e^{-i\alpha_5 N_3} e^{-i\alpha_6 M_3} \; .$$

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